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NOTES ON AXIOMS FOR QUANTUM MECHANICS

by

M. D. MacLaren

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M. D. MacLaren

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PREFACE

This set of notes is an expansion of two lectures given at Argonne as part of a continuing seminar on the foundations of quantum theory. In the notes, we attempt to survey, from a mathematical point of view, the problem of giving a precise and attractive set of axioms for nonrelativistic quantum mechanics and to point out some possibilities for future research.

While this problem is not one of the central problems in physics, it has been of interest since the 1930's, and in recent years quite a bit of work has been done on the subject. At present, the problem is in no sense solved, and which line of attack on it is most promising is not at all clear. As a result, these notes are rather informal and contain certain remarks that the author may well wish to retract at some future time. The list of references is intended to give reasonably complete coverage of the mathematical papers on the subject, and the author would appreciate being notified of any omissions.

Drs. Joe Cook and Joe Moyal have kindly looked over the notes and pointed out several errors. However, they are not responsible for any that remain.

I. INTRODUCTION

Consider a pair of objects $(\mathcal{O},\mathcal{S})$, where $\mathcal{O}$ is to represent the set of observables of some physical system and $\mathcal{S}$ the set of states. Suppose that we are dealing with the quantum mechanics of a system with a finite number of degrees of freedom. Then the theory can be based on a complex separable Hilbert space H. In this case, $\mathcal{O}$ is $\mathcal{O}(H)$, the set of all self-adjoint operators on H, and $\mathcal{S}$ is $\mathcal{S}(H)$, the set of all nonnegative self-adjoint operators with trace one. (Here we are including the mixed states in $\mathcal{S}$.) For each element A in $\mathcal{O}(H)$ and F in $\mathcal{S}(H)$, we can define a probability measure μ on the real line by

$$\mu(E) = \text{Trace}\,[F\chi_E(A)],$$

where E is a measurable set.* The physical interpretation of this is that $\mu(E)$ is the probability that a measurement of A in the state F will result in a value in the set E. The expected value of A, if it exists, is just Trace (FA).

*Here χ_E is the characteristic function of the set E, and $\chi_E(A)$ is the corresponding operator constructed from A by means of the functional calculus. If E_λ is the spectral resolution of A,

$$\chi_E(A) = \int_E \lambda \, dE_\lambda.$$

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$$\chi_E(A) = \int_E d E_\lambda .$$

Now the above paragraph gives an axiomatization of the pair $(\mathcal{O},\mathcal{S})$ that is both precise and concise. However, two objections to the axiomatization may be raised. First, in this form, the axioms do not suggest natural generalizations; yet some generalization is no doubt needed to handle relativistic problems. Second, the axioms are quite "ad hoc." Much better would be a larger set of simple axioms, each one representing some one physical or mathematical principle. The subject of this set of notes is the problem of finding such a set of axioms, and also of what one can say about the pair $(\mathcal{O},\mathcal{S})$ when it has some, but not all, of the properties of $[\mathcal{O}(H), \mathcal{S}(H)]$.

This general problem has attracted a fair amount of attention over the years without any completely satisfactory results being produced. This may be because the mathematical problems involved are very difficult; but it is rather more likely that they are simply unfamiliar and outside the main stream of mathematics. So that the reader may form his own judgment on this, a list of references is included. It is hoped that this list will give a fair picture of the mathematical literature on this subject, but no guarantees on this are extended. No doubt there are also many papers in the physics literature on this subject, but we have not attempted to survey them. This may not be a serious omission, for what appears to be lacking in this subject is not so much physical ideas as mathematical results.

To give some form to the notes, we will aim at presenting a set of axioms, from which it can be deduced that $(\mathcal{O},\mathcal{S})$ is $[\mathcal{O}(H), \mathcal{S}(H)]$. We attempt to give exact statements of the important theorems, but proofs have been omitted. They are either trivial or may be found in the literature. Possibilities for generalizations, alternate axiom schemes, results using only a few axioms, etc., will be discussed as we go along. However, we make no claim to complete coverage of known results.

The set of axioms that we do give derives primarily from the "quantum logic" approach first set forth by Birkhoff and von Neumann,[1] and lately developed by Mackey.[10,11] However, there is some flavor of the "algebraic" approach initiated by Jordan, developed by Jordan, von Neumann, and Wigner,[8,21] and lately emphasized by Segal.[14,15] In fact, one thing we try to do is relate the problems arising in the two approaches.

One point should be mentioned in passing. Nothing has been said about any group acting on $(\mathcal{O},\mathcal{S})$; yet that is an important part of physics. The reason for this is that we simply do not see how the group affects the representation of $(\mathcal{O},\mathcal{S})$. However, it may well be that no satisfactory axiomatization of the system $(\mathcal{O},\mathcal{S})$ can be obtained without considering the group that acts on it.

II. AXIOMS FOR QUANTUM LOGIC

In this section, following Mackey,[11] we present five basic and rather elementary axioms for the pair $(\mathcal{O},\mathcal{S})$. A consequence of these axioms will be that the study of $(\mathcal{O},\mathcal{S})$ reduces to the study of $(\mathcal{Q},\mathcal{S}^*)$, where $\mathcal{Q}$ is the set of observables taking on only the values zero and one, and $\mathcal{S}^*$ is the set of states looked at as acting only on $\mathcal{Q}$.

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Axiom 1. Each element $\phi \varepsilon S$ is a function from $\mathcal{O}$ to the set of all Borel probability measures on the real line.

For $A \varepsilon \mathcal{O}$, $\phi \varepsilon S$, and E a Borel set, we let $\phi^A(E)$ denote the measure of E. As a suggestive notation, we may write $\text{Prob}(A \varepsilon E | \phi)$ for $\phi^A(E)$. The physical interpretation of Axiom 1 is just that $\phi^A(E)$ is the probability that a measurement of A in the state ϕ will give a result in the set E. Thus Axiom 1 is really just the definition of "state."

Axiom 2. If A and B are in $\mathcal{O}$, and $\phi^A(E) = \phi^B(E)$ for all ϕ in S and all E, then $A = B$. If ϕ and ψ are in S, and $\phi^A(E) = \psi^A(E)$ all A in $\mathcal{O}$, all E, then $\phi = \psi$.

Axiom 3. Let $A \varepsilon \mathcal{O}$, and let f be a real-valued Borel measurable function on the real line. Then there exists B in $\mathcal{O}$ such that $\phi^B(E) = \phi^A[f^{-1}(E)]$ for all ϕ in S and all measurable sets E.

<u>Proposition 1.</u> B is uniquely determined by A.

We will denote the observable B by f(A). The physical interpretation of f(A) is just that one measures A, getting a result, say λ, and then computes $f(\lambda)$. It follows from Axiom 3 that there exists for each real λ a constant observable taking on only the value λ; and for convenience, we will let λ denote this observable. Also, for any observable A and measurable set E, we have the observable $Q = \chi_E(A)$. It is clear that Q is an especially simple sort of observable, taking on only the values zero and one. We call such an observable a *question*. The term *proposition* is also used at times. The set of all questions Q in $\mathcal{O}$ will be denoted by $\mathcal{Q}$.

<u>Proposition 2.</u> Q is a question if and only if $Q^2 = 1$.

For an observable A and state ϕ, let $m_\phi(A)$ denote the mean value of A in the state ϕ, if it exists. Obviously, $m_\phi(Q)$ exists for all questions Q. The set of functions m_ϕ, ϕ in S, mapping $\mathcal{Q}$ into the unit interval, will be denoted by S^*. The functions m in S^* define a partial ordering on $\mathcal{Q}$ in an obvious way; namely, $Q_1 \leq Q_2$ *if and only if* $m(Q_1) \leq m(Q_2)$ *all* m *in* S^*.

<u>Proposition 3.</u> The relation $\leq$ is a partial ordering on the set of all questions.

<u>Proposition 4.</u> There exist questions zero and one such that $m(0) = 0$ and $m(1) = 1$ all $m \varepsilon S$. Thus $0 \leq Q \leq 1$ all $Q \varepsilon \mathcal{Q}$. (These are just the constant observables zero and one, which exist by Axiom 3.)

<u>Proposition 5.</u> If Q is a question, then $(1 - Q)$ is a question. ($1 - Q$ exists by Axiom 3.)

Let $\{Q_\alpha\}$ be any set of questions. The question P is said to be a least upper bound for $\{Q_\alpha\}$ if $P \geq Q_\alpha$ for all α and if $R \geq Q_\alpha$ for all α implies $R \geq P$. The term greatest lower bound is defined analogously. We write $Q_1 \cup Q_2$ and $Q_1 \cap Q_2$ for the least upper bound and greatest lower bound, respectively, providing they exist. Now it is not necessarily true that every set $\{Q_\alpha\}$, or even every pair $\{Q_1, Q_2\}$, has a least upper bound or greatest lower bound. When every pair in a partially ordered set has

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Theorem 1. *Write* $Q' = 1 - Q$ *for* Q *in* $\mathcal{Q}$. *Then the mapping* $Q \to Q'$ *is an orthocomplementation on the partially ordered set* $\mathcal{Q}$; *i.e.,*

1) *If* $Q_1 \leq Q_2$, *then* $Q_1' \geq Q_2'$;

2) $Q'' = Q$;

3) $Q \cup Q'$ *and* $Q \cap Q'$ *exist and equal one and zero, respectively.*

The orthocomplementation leads to a notion of orthogonality which is quite analogous to that in a Hilbert space. We say that two questions Q_1 and Q_2 are *orthogonal*, and write $Q_1 \perp Q_2$, if $Q_1 \leq Q_2'$.

Proposition 6. The orthogonality relation is symmetric.

The essential physical interpretation of orthogonality is that we consider two orthogonal questions Q_1 and Q_2 to be simultaneously measurable. That being the case, we naturally assume that there exist observables of the form $\lambda_1 Q_1 + \lambda_2 Q_2$, λ_i real numbers. Now we can give examples of mutually orthogonal projections by the following construction. Let χ_E denote the characteristic function of the Borel set E, and let A be an observable. Then $\chi_E(A)$ is a question, and $1 - \chi_E(A) = \chi_{E'}(A)$ (E' is the complement of E). Thus, if E and F are disjoint sets, $\chi_E(A)$ and $\chi_F(A)$ are orthogonal questions. Moreover, if $\{E_i\}$ is a countable family of disjoint sets and $F = \cup E_i$, we have $m[\chi_F(A)] = \Sigma m[\chi_{E_i}(A)]$ for all m in S^*. It is reasonable to say that the question $\chi_F(A)$ is the sum of the $\chi_{E_i}(A)$. With these considerations in mind (and further discussion may be found in Mackey's article[10] or book[11]), we introduce:

Axiom 4. *Let* $\{Q_j\}$ *be a pairwise orthogonal sequence of questions. Then there exists a question* P *such that* $m(P) = \Sigma m(Q_j)$ *for all* m *in* S^*.

Theorem 2. *(Kadison) If* $\{Q_j\}$ *is a pairwise orthogonal set of questions and* $R \geq Q_j$ *for all* j, *then* $R \geq \Sigma Q_j$. *Thus* ΣQ_j *is the least upper bound to the set* $\{Q_j\}$.

Theorem 3. *If* $Q_1 \leq Q_2$, *then there exists a question* P *such that* $m(P) = m(Q_2) - m(Q_1)$ *for all* m *in* S^*. *Moreover,* $P = Q_1' \cap Q_2$, *and* $Q_2 = P \cup Q_1$.

Proof. This follows from Axiom 4 and Theorem 2. We set $P = 1 - (Q_2' + Q_1)$.

An orthocomplemented partially ordered set such that $Q \leq P$ implies $P = Q \cup (Q' \cap P)$ is said to be *weakly modular*. *Relatively orthocomplemented* would be another good term.

Now we are in a position to introduce, within the quantum logic framework, the notions corresponding to observable and state. A *question-valued measure* is a function $E \to Q_E$ from the Borel measurable sets of the real line to $\mathcal{Q}$ having the following properties:

both a greatest lower bound and least upper bound, the set is said to be a lattice. With only the axioms given so far, Q need not be a lattice. However, it does have some structure beyond the partial ordering.

<u>Theorem 2.</u> *Write $Q' = 1 - Q$ for Q in Q. Then the mapping $Q \to Q'$ is an orthocomplementation on the partially ordered set Q, i.e.,*

1) If $Q_1 \leq Q_2$, then $Q_2' \leq Q_1'$

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<u>Theorem 2.</u> (Mackey) *If $\{Q_i\}$ is a pairwise orthogonal set of questions and $R \geq Q_i$ for all i, then $R \geq \Sigma Q_i$. Thus ΣQ_i is the least upper bound to the set $\{Q_i\}$.*

<u>Theorem 3.</u> *If $Q_1 \leq Q_2$, then there exists a question P such that $m(P) = m(Q_2) - m(Q_1)$ for all m in S^*. Moreover, $P = Q_1' \cap Q_2$, and $Q_2 = P \cup Q_1$.*

<u>Proof.</u> This follows from Axiom 4 and Theorem 2. We set $P = 1 - (Q_2' + Q_1)$.

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i) $E \cap F = \phi$ implies $Q_E \perp Q_F$;

ii) $E_i \cap E_j = \phi$, $i \neq j$, implies $Q_{\cup E_i} = \Sigma Q_{E_i}$;

iii) $Q_\phi = 0$, $Q_{(-\infty,\infty)} = 1$.

A *measure on the questions* is a positive real-valued function m on $\mathcal{Q}$ such that $m(0) = 0$, $m(1) = 1$, and $m(\Sigma Q_j) = \Sigma m(Q_j)$ for any orthogonal sequence of questions $\{Q_j\}$.

A question-valued measure Q_E may be regarded as the quantum mechanical generalization of a random variable. For each measure on the questions m, the mapping $E \to m(Q_E)$ is an ordinary probability measure. Thus a question-valued measure is a whole family of probability measures, which can be related in very complicated ways.

Every observable A in $\mathcal{O}$ has an associated question-valued measure $E \to X_E(A)$, and is, in fact, completely determined by it, for we have $\mathrm{Prob}(A \varepsilon E|\phi) = m_\phi[X_E(A)]$ for all ϕ in S. Moreover, this last expression depends only on the action of m on $\mathcal{Q}$, that is, on the function m_ϕ in S^*. We may summarize all this in the following proposition.

Proposition 7. The observable A is completely determined by its associated question-valued measure $E \to X_E(A)$. The state ϕ is completely determined by the function $m_\phi : Q \to m_\phi(Q)$, which is a measure on the questions.

Thus we see that the whole structure of $\mathcal{O}$ and S is almost determined by $\mathcal{Q}$ and the measures on the questions m in S^*. It would be completely determined if we only knew which question-valued measures corresponded to observables, i.e., which are of the form $E \to X_E(A)$.† The mathematician's answer to this problem is easy; we introduce another axiom.

Axiom 5. For every question-valued measure, $E \to Q_E$, there exists an observable A in $\mathcal{O}$ such that $Q_E = X_E(A)$ for all measurable sets E.

With the introduction of this axiom, the study of $(\mathcal{O},S)$ is reduced to the study of the "quantum logic" $(\mathcal{Q},S^*)$. The essential features of such a logic are these: the set $\mathcal{Q}$ is a weakly-modular, orthocomplemented, partially-ordered set, in which every countable orthogonal subset of $\mathcal{Q}$ has a least upper bound. The set S^* is a family of measures on $\mathcal{Q}$ (the term "measure" is defined here as we defined "measure on the questions" above) large enough so that $Q_1 \leq Q_2$ if and only if $m(Q_1) \leq m(Q_2)$ for all m in S^*. The idea of looking at such a logic was first put forth, in a rather different form, by Birkhoff and von Neumann.[1] The development here is from pages 61-68 of Mackey.[11] Quantum logics have also been studied recently by Varadarajan[19] and Pool.[13] Authors who have studied logics with the additional axiom that $\mathcal{Q}$ is a lattice are Zierler,[22] Piron,[12] Emch and Piron,[3] and Gudder.[7]

Within the quantum logic framework we can introduce two familiar concepts: simultaneous observables and the spectrum. It was mentioned

†Note that we have not assumed that every measure on the questions is in S^*. For the special pair $[\mathcal{O}(H), S(H)]$, this turns out to be the case, an important theorem proved by Gleason.[6]

i) $E \cap F = \phi$ implies $Q_E \perp Q_F$;

ii) $E_i \cap E_j = \phi$, $i \neq j$, implies $Q_{\cup E_i} = \Sigma Q_{E_i}$;

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A measure on the questions is a positive real-valued function m on $\mathcal{Q}$ such that $m(0) = 0$, $m(1) = 1$, and $m(\Sigma Q_j) = \Sigma m(Q_j)$ for any orthogonal sequence of questions $\{Q_j\}$.

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Proposition 7. The observable A is completely determined by its associated question-valued measure $E \to X_E(A)$. The state ϕ is completely determined by the function $m_\phi:Q \to m_\phi(Q)$, which is a measure on the questions.

Thus we see that the whole structure of $\mathcal{O}$ and S is almost determined by $\mathcal{Q}$ and the measures on the questions m in S^*. It would be completely determined if we only knew which question-valued measures corresponded to observables, i.e., which are of the form $E \to X_E(A)$.* The mathematician's answer to this problem is easy; we introduce another axiom.

Axiom 6. For every question-valued measure, $E \to Q_E$, there exists an observable A in $\mathcal{O}$ such that $Q_E = X_E(A)$ for all measurable sets E.

With the introduction of this axiom, the study of $(\mathcal{O},S)$ is reduced to the study of the "quantum logic" $(\mathcal{Q},S^*)$. The essential features of such a logic are these: the set $\mathcal{Q}$ is a weakly-modular, orthocomplemented, partially-ordered set, in which every countable orthogonal subset of $\mathcal{Q}$ has a least upper bound. The set S^* is a family of measures on $\mathcal{Q}$ (the term "measure" is defined here as we defined "measure on the questions" above) large enough so that $Q_1 \leq Q_2$ if and only if $m(Q_1) \leq m(Q_2)$ for all m in S^*. The idea of looking at such a logic was first put forth, in a rather different form, by Birkhoff and von Neumann.[4] The development here is from pages 61-68 of Mackey.[11] Quantum logics have also been studied recently by Varadarajan[19] and Pool.[13] Authors who have studied logics with the additional axiom that $\mathcal{Q}$ is a lattice are Zierler,[23] Piron,[12] Emch and Piron,[8] and Gudder.[7]

Within the quantum logic framework we can introduce two familiar concepts: simultaneous observables and the spectrum. It was mentioned

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*Note that we have not assumed that every measure on the questions is in S^*. For the special pair $[\mathcal{O}(\mathcal{H}), S(\mathcal{H})]$, this turns out to be the case, an important theorem proved by Gleason.[9]

above that orthogonal questions were assumed to be simultaneously measurable in the laboratory. Also this follows from Axiom 5, for then both questions are functions of a single observable. The general definition of commuting observables is based on this, and two questions Q and P are said to *commute* if there exist orthogonal questions R_1, R_2, and R_3 such that $Q = R_1 + R_2$, and $P = R_1 + R_3$. Two observables A and B *commute* if $\chi_E(A)$ and $\chi_F(B)$ commute for all measurable E and F.

The spectrum of an observable A is physically just the set of all possible values that one may get for a measurement of A. This can be made precise by letting the *spectrum* of an observable A be the closed set Sp(A) which is the complement of the union of all open sets E such that $\text{Prob}(A\varepsilon E|\phi) = 0$ for all ϕ in S.

From the spectrum, we can define bounds for observables. Let A be an observable. The *norm* of A is $\|A\| = \sup |\lambda| [\lambda\varepsilon\text{Sp}(A)]$. A is *bounded* if $\|A\| < \infty$. We also define lower and upper bounds for A:

$$\|A\|_- = \inf \lambda[\lambda\varepsilon\text{Sp}(A)], \quad \|A\|_+ = \sup \lambda[\lambda\varepsilon\text{Sp}(A)].$$

Now one can spend quite a bit of time discussing these various concepts without introducing any further axioms; but any significant development of the theory appears impossible without more axioms. We should note that one can formulate in this abstract setting, a noted theorem of von Neumann, namely, that for a countable family of commuting observables $\{A_n\}$, there exists an observable B and measurable functions $\{f_n\}$ such that $A_n = f_n(B)$ for all n. Varadarajan almost proved this theorem in Ref. 19. However, Pool[13] pointed out that a hidden assumption was made, namely, that if Q_1, Q_2, and Q_3 commute, then Q_3 commutes with $Q_1 + Q_2$. Pool shows that this does not necessarily hold and gives various equivalent and sufficient conditions. In particular, he shows that the necessary condition does hold if $\mathcal{Q}$ is a lattice.

III. THE ALGEBRAIC APPROACH

In this section, we briefly discuss an alternative approach to the problem of finding axioms for quantum mechanics. This goes back to Jordan, who considered the question, "What algebraic operations are meaningful in the set of observables?" Let A denote the set of all bounded observables, which is the natural object of study in the algebraic approach. Jordan noted that in A one has available a sum, A + B, multiplication by real numbers, and also powers, i.e., A^2, A^3, etc. No product in the ordinary sense exists in A, for the product of two self-adjoint operators is not in general self-adjoint. However, the symmetric product $A \circ B = \frac{1}{2}(AB + BA)$ is in A, and, moreover, this may be defined using only sums and squares; i.e.,

$$A \circ B = \tfrac{1}{2}[(A + B)^2 - A^2 - B^2].$$

An algebra with this sort of product is now called a Jordan algebra.†

†Strictly speaking, a Jordan algebra is a nonassociative algebra in which the following identity holds: $(a^2b)a = (ab)a^2$, where $a^2 = aa$. The algebra may be over any field, but only the field of real numbers is of physical interest.

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Jordan, von Neumann, and Wigner[8] worked out the theory of real Jordan algebras having a finite linear basis and satisfying the condition that $a^2 + b^2 + \ldots = 0$ implies $a = b = \ldots = 0$. It turns out that every such algebra is the direct sum of irreducible algebras, and they classified all the irreducible algebras. Apart from two exceptional cases, the irreducible algebras are just the algebras of all n x n Hermitian matrices over the real, complex, or quaternionic numbers. One exceptional case is the set of algebras in which the maximal number of orthogonal idempotents is two. The dimension of such an algebra is arbitrary, but they are easily described; they appear to have no physical interest. The other exceptional case is an algebra of 3 x 3 matrices whose elements are Cayley numbers.

Von Neumann[21] started to extend the ideas and results of Ref. 8 to the case in which the algebra does not have a finite linear basis. This, of course, is the situation for the algebra of quantum mechanical observables. Naturally, to replace the finiteness condition, it was necessary to introduce certain topological assumptions. Unfortunately, the second part of his paper was never published and did not appear in his files. About all we can say at this time about the structure of infinite dimensional Jordan algebras, is that there exists a remarkable variety of such algebras, even restricting consideration to those which are algebras of self-adjoint operators. A complete classification of infinite-dimensional Jordan algebras, or even a significant characterization of those that are, say, weakly-closed algebras on Hilbert space, must be very difficult to find. However, it is not unreasonable to suppose that the ideas and methods of Refs. 8 and 21 could be extended to give a characterization of the special algebra of all bounded self-adjoint operators.

More recently, Segal[14,15] has adopted a similar algebraic approach to the foundations of quantum mechanics. He focuses attention on the set A of bounded observables, and assumes that they form a complete normed vector space over the real numbers. In addition, he supposes that there exists a unit element one, and that for each observable A, and each positive integer n, a bounded observable A^n is defined in such a way that the normal rules for calculation with polynomials of a single variable are satisfied. Finally, he introduces three further postulates relating the norm and the squaring operation:*

S1) $\|U^2 - V^2\| \leq \mathrm{Max}[\|U^2\|, \|V^2\|]$;

S2) $\|U^2\| = \|U\|^2$;

S3) U^2 is a continuous function of U.

From these postulates, he is able to prove several rather general results about spectra and states. (For Segal, a state is a bounded positive linear functional E such that E(1) = 1.)

We quote the following results from pages 6 and 7 of Ref. 14:

1. *"There exists an ample supply of pure states, in the sense that two observables having the same expectation values in all pure states must be identical."*

*We have omitted one postulate shown to be redundant by Sherman.[16]

Jordan, von Neumann, and Wigner[8] worked out the theory of real Jordan algebras having a finite linear basis and satisfying the condition that $a^2 + b^2 + \ldots = 0$ implies $a = b = \ldots = 0$. It turns out that every such algebra is the direct sum of irreducible algebras, and they classified all the irreducible algebras. Apart from two exceptional cases, the irreducible algebras are just the algebras of all $n \times n$ Hermitian matrices over the real, complex, or quaternionic numbers. One exceptional case is the set of algebras in which the maximal number of orthogonal idempotents is two. The dimension of such an algebra is arbitrary, but they are easily described; they appear to have no physical interest. The other exceptional case is an algebra of 3×3 matrices whose elements are Cayley numbers.

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1. "There exists an ample supply of pure states, in the sense that two observables having the same expectation values in all pure states must be identical."

*We have omitted one postulate shown to be redundant by Sherman.[18]

2. *"Any observable admits a closed set of spectral values, and the expectation of the observable in any state is the average of these spectral values with respect to a probability distribution on them canonically determined by the state."*

3. *"The smallest closed system of observables (in the sense of the phenomenological postulates) containing a given observable A is in 1-1 algebraic correspondence with the algebra of all continuous functions on the spectrum of A."*

4. *"Any pure state of a physical system which is a subsystem of a larger system can be realized in a pure state of the larger system."*

5. *"The bound of an observable A may be defined purely algebraically as the least real number α such that $I - A = B^2$ and $\alpha I + A = C^2$ for suitable observables B and C."*

These results were proved in Ref. 15, except for (4), for which the additional assumption was made that the sum of squares was always a square. However, that follows from the axioms (Sherman[17]).

To continue his development of quantum mechanics beyond these rather general abstract results, Segal assumes that A is the set of all self-adjoint elements in a C* algebra. This corresponds to the "ad hoc" assumption that $\mathcal{O}$ is $\mathcal{O}(H)$. Of course, what Segal is primarily interested in is which particular C* algebras are suitable for quantum mechanics and the further development of the theory from that point. On the other hand, the problem we are discussing here is how to deduce from elementary axioms that $\mathcal{O}$ is $\mathcal{O}(H)$ or, more generally, that the bounded observables are the self-adjoint elements in a C* algebra.

It is worth pointing out that there is a variety of A's satisfying Segal's postulates but not coming from a C* algebra. Sherman[16] and Lowdenslager[9] give whole familes of examples where the distributive law, $A\circ(B + C) = A\circ B + A\circ C$, does not hold for the Jordan product. Sherman also shows that the exceptional Jordan algebra of 3 x 3 matrices of Cayley numbers satisfies Segal's postulates. Finally, and perhaps most interesting, there are weakly-closed Jordan algebras of self-adjoint operators on Hilbert space which are not the set of self-adjoint operators in a C* algebra. (See Topping.[18])

To summarize, the essential feature of the algebraic approach is the assumption that the sum of bounded observables exists, but as we shall see, this may also be applied in the quantum logic framework. The difficulties in the algebraic approach are the lack of any physical reason for assuming that the special product is distributive, and the absence of any representation theorems for infinite-dimensional Jordan algebras.

IV. THE LATTICE PROPERTY FOR QUANTUM LOGICS

In order to get any real development of the theory of quantum logics, it seems necessary to know that $\mathcal{Q}$ is a lattice. Because $\mathcal{Q}$ is orthocomplemented, this will follow if every pair of questions P,Q has a greatest

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In order to get any real development of the theory of quantum logics, it seems necessary to know that Q is a lattice. Because Q is orthocomplemented, this will follow if every pair of questions P,Q has a greatest lower bound $P \cap Q$. This is often taken as an axiom, e.g., by Piron,[12] Gudder,[7] and Zierler.[23] However, the question arises as to what physical justification there is for such an axiom. The above authors skip over this point, but Birkhoff has discussed it in Ref. 2. In order to give Birkhoff's arguments for assuming the existence of greatest lower bounds, it will be necessary to digress a bit and briefly describe his approach to quantum logic. This is appropriate in any case, for it is the original approach of Birkhoff and von Neumann.[1]

In our presentation, questions are assumed to correspond directly to measurements. For Birkhoff, on the other hand, a proposition is a prediction with probability one about the result of an experiment. To connect the two notions, we will use the notation $\bar{Q}$ to stand for the prediction that a measurement of the question Q will give the result one with certainty. The states ϕ for which $\bar{Q}$ is a true prediction are just those for which $m_\phi(Q) = 1$. Note that in some states ϕ, the result of measuring Q may be one, but not with certainty. These are just the states for which $0 < m_\phi(Q) < 1$, and it is perhaps reasonable to say that in such states $\bar{Q}$ is neither true nor false. Now there is a natural ordering of these predictions; namely, $\bar{Q} \leq \bar{P}$ if and only if $\bar{Q}$ implies $\bar{P}$, which means, in terms of the questions Q and P, that for every state ϕ such that $m_\phi(Q) = 1$, $m_\phi(P) = 1$ must hold. This ordering of implication is just that used to order the propositions of classical logic, so that it is more reasonable to call $\bar{Q}$ a proposition than it is to call Q one.

Now Birkhoff suggests defining $\bar{P} \cap \bar{Q}$ as the prediction that measurements of both P and Q are certain to give the result one. Because P and Q do not in general commute, the concept of measuring both must be made precise. One possibility is to measure P and then immediately afterwards measure Q. Suppose we denote the prediction that both measurements are one by $\bar{P} \cap \bar{Q}$, and the prediction for the measurements made in the reverse order by $\bar{Q} \cap \bar{P}$. Then the question arises as to whether or not $\bar{P} \cap \bar{Q} = \bar{Q} \cap \bar{P}$. But this question can be tested experimentally! If the equality $\bar{P} \cap \bar{Q} = \bar{Q} \cap \bar{P}$ can be experimentally verified, we have good reason for making the existence of greatest lower bounds one of the properties of the logic of predictions.

In discussion with the author, Birkhoff has also suggested another way of looking at the prediction $\bar{P} \cap \bar{Q}$. In quantum mechanics, one always assumes that experiments are reproducible, which implies the existence of an unlimited supply of similar systems all in the same state ϕ. Thus, to observe the result of measuring P and Q in the state ϕ, it is not necessary to make the measurements on the same system. We can measure P for one system, then Q for a second, then P for a third, etc. As a result of this process we get a picture of the distribution of the results of both P and Q in the state ϕ, without any interference between the different measurements. The proposition $\bar{P} \cap \bar{Q}$ is considered to be true for ϕ if all the measurements give one.

Now the above arguments justify assuming the existence of greatest lower bounds in the logic of predictions. Is this a justification for assuming their existence in $\mathcal{Q}$, the set of questions? This is a rather tricky point, but the answer seems to be "No." At least the definition of $\bar{Q} \cap \bar{P}$ does not yield what one would normally call an operational definition lower bound $\tilde{P} \cap \tilde{Q}$. This is often taken as an axiom, e.g., by Piron,[17] Gudder,[7] and Zierler.[19] However, the question arises as to what physical justification there is for such an axiom. The above authors skip over this point, but Birkhoff has discussed it in Ref. 2. In order to give Birkhoff's arguments for assuming the existence of greatest lower bounds, it will be necessary to digress a bit and briefly describe his approach to quantum logic. This is appropriate in any case, for it is the original approach of Birkhoff and von Neumann.[3]

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An obvious idea for measuring $P \cap Q$ is to measure P and then Q, but this will not correspond to measuring $P \cap Q$, and it is perhaps worthwhile working out what actually happens in the conventional theory, i.e., when $(\mathcal{O},\mathcal{S})$ is $[\mathcal{O}(\mathcal{H}), \mathcal{S}(\mathcal{H})]$. Suppose we measure P then Q and define the compound measurement R to be one only if both P and Q are one, and zero otherwise. (Here we are using the same letter for a question Q, which is a projection in the Hilbert space $\mathcal{H}$, and the corresponding laboratory measurement.) Suppose the system is initially in a state given by the unit vector x. Then the probability that P will be one is (Px,x). Given that P is one, the state after the measurement is given by the unit vector $(Px)/\|Px\|$. Hence the probability that Q will be one, given that P was one, is $(QPx,Px)/\|Px\|^2$. Therefore the probability that R will be one is

$$(QPx,Px)(Px,x)/\|Px\|^2 = (PQPx,x).$$

Clearly this probability is not independent of the order in which P and Q are measured unless P and Q commute. Thus R cannot be a measurement corresponding to $\bar{P} \cap \bar{Q}$. Moreover, R, although it can be measured by an operationally defined experiment, is not an observable. That is, there is no question R in $\mathcal{O}(\mathcal{H})$ corresponding to the measurement. If there were, we would have $(Rx,x) = (PQPx,x)$ for all x, which means $R = PQP$; but PQP is not a projection unless P and Q commute.

The procedure of measuring P, then Q, then P, etc., can be carried out indefinitely, at least as a Gedankenexperiment. The result is rather interesting. Let R_n be the measurement corresponding to measuring P, Q, P, etc., with a total of n measurements. Then the probability that R_n is one turns out to be $(PQP\ldots QPx,x)$, with a total of $2n - 1$ factors. But in the strong operator topology, the sequence P, PQP, ... , approaches the limit $P \cap Q$. Thus we do not have an operational procedure for measuring $P \cap Q$ but only approximations to one, and the approximation cannot be made uniform over all states ϕ, because the convergence is only in the strong topology. Moreover, the approximate measurements are not observables in the conventional theory.

We might summarize the above discussion by saying that $P \cap Q$ is not an operational concept but is close to being one. This is at least better than the situation in regard to the sum of two observables. Nevertheless, our own analysis to show that $\mathcal{Q}$ is a lattice will be based on the assumption that the sum of any two bounded observables exists. It is interesting to connect up the algebraic and quantum logic approaches in this way, and also we feel that the sum axiom may be very useful at other points. Before we give the sum axiom, it is appropriate to introduce another important axiom, which fortunately does have a physical justification. It corresponds of $Q \cap P$. One of the physical assumptions underlying our axioms is that for each question Q, in fact for each observable, there is a corresponding experimental procedure or measurement process. This measurement can be carried out on a single physical system and, moreover, if the result of a measurement of Q is one, then the system is supposed immediately after the measurement to be in a state s such that $s(Q) = 1$. But given measuring procedures for P and Q, there does not seem to be any way to describe a measuring process for $P \cap Q$; i.e., there is apparently no operational definition of $P \cap Q$.

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$$(QPx,Px)(Px,x)/\|Px\|^2 = (PQPx,x).$$

Clearly this probability is not independent of the order in which P and Q are measured unless P and Q commute. Thus R cannot be a measurement corresponding to $P \cap Q$. Moreover, R, although it can be measured by an operationally defined experiment, is not an observable. That is, there is no question R in $Q(H)$ corresponding to the measurement. If there were, we would have $(Rx,x) = (PQPx,x)$ for all x, which means $R = PQP$; but PQP is not a projection unless P and Q commute.

The procedure of measuring P, then Q, then P, etc., can be carried out indefinitely, at least as a Gedankenexperiment. The result is rather interesting. Let R_n be the measurement corresponding to measuring P, Q, P, etc., with a total of n measurements. Then the probability that R_n is one turns out to be $(PQP\ldots QPx,x)$, with a total of $2n - 1$ factors. But in the strong operator topology, the sequence P, PQP, ... , approaches the limit $P \cap Q$. Thus we do not have an operational procedure for measuring $P \cap Q$ but only approximations to one, and the approximations cannot be made uniform over all states s, because the convergence is only in the strong topology. Moreover, the approximate measurements are not observables in the conventional theory.

We might summarize the above discussion by saying that $P \cap Q$ is not an operational concept but is close to being one. This is at least better than the situation in regard to the sum of two observables. Nevertheless, our own analysis to show that Q is a lattice will be based on the assumption that the sum of any two bounded observables exists. It is interesting to connect up the algebraic and quantum logic approaches in this way, and also we feel that the sum axiom may be very useful at other points. Before we give the sum axiom, it is appropriate to introduce another important axiom, which fortunately does have a physical justification. It corresponds to the physical assumption that the measurement of a question repeated immediately will give the same result.

Axiom 6. *For every question* Q, *there exists a state* ϕ *such that* $m_\phi(Q) = 1$.

This axiom lets one relate the spectrum of an observable to its mean values. (Zierler[22] has discussed this.) In fact, we have:

Proposition 8. Let A be an observable. Then $\lambda \varepsilon Sp(A)$, if and only if for all $\varepsilon > 0$, there exists ϕ in S such that $|m_\phi(A) - \lambda| < \varepsilon$. Hence,

$$||A||_- = \inf m_\phi(A) \text{ all } \phi \varepsilon S;$$

$$||A||_+ = \sup m_\phi(A) \text{ all } \phi \varepsilon S;$$

$$||A|| = \sup |m_\phi(A)| \text{all } \phi \varepsilon S.$$

The next axiom is perhaps the key to any complete development of quantum mechanical axiomatics along algebraic lines.

Axiom 7. *Let* A *and* B *be bounded observables. Then there exists a unique observable* $C = A + B$ *such that* $m_\phi(C) = m_\phi(A) + m_\phi(B)$ *for all* $\phi \varepsilon S$.

This axiom is simple and widely accepted as being basic. There is, however, no obvious physical justification for it. While characteristic of the Jordan algebra approach to axiomatics, it may also be exploited in studying the quantum logic. From Proposition 8, we immediately ge the following result.

Proposition 9. Let A and B be bounded observables. Then

$$||A + B||_- \geq ||A||_- + ||B||_-,$$

$$||A + B||_+ \leq ||A||_+ + ||B||_+,$$

and

$$||A + B|| \leq ||A|| + ||B|| .$$

This shows that the set of bounded observables is a normed vector space over the real numbers. In fact, it satisfies most of Segal's axioms, the only question being about completeness in the norm and continuity of the squaring operation.

Now suppose that P and Q are questions, and let $A = P + Q$, $R = X_{\{2\}}(A)$. By the above proposition, $||A||_- = 0$, $||A||_+ = 2$. Thus, if $m_\phi(A) = 2$ we must have $m_\phi(R) = 1$. From this we conclude that $m_\phi(R) = 1$ if and only if $m_\phi(P) = m_\phi(Q) = 1$. This suggests that the question R is the greatest lower bound for P and Q. We apparently cannot prove this by using only Axioms 1-7, so we introduce:

Axiom 8. *Suppose that* P *and* Q *are questions such that* $m_\phi(P) = 1$ *implies* $m_\phi(Q) = 1$. *Then* P *and* Q *commute.*

It follows immediately from this axiom that if $m_\phi(P) = 1$ implies $m_\phi(Q) = 1$ for all states ϕ, then $P \leq Q$. Thus we have, in effect, assumed that the ordering of implication, which we discussed in connection with the existence of $P \cap Q$, is identical to the ordering by mean values. Zierler[22] has shown that if the logic is a lattice, then Axiom 8 holds. Here we go the other way, and using Axioms 6, 7, and 8, show that $\mathcal{Q}$ is a lattice. Applying Axiom 8 to $R = \chi_{\{2\}}(P + Q)$, we see that $R \leq P$ and $R \leq Q$. Thus R is a lower bound for P and Q. On the other hand, if $S \leq P$ and $S \leq Q$, then $m_\phi(S) = 1$, implies $m_\phi(P + Q) = 2$, which implies $m_\phi(R) = 1$, so that $S \leq R$. Thus we have proved that every pair of questions P,Q has a greatest lower bound. Since $\mathcal{Q}$ is orthocomplemented, this means that $\mathcal{Q}$ is a lattice and we have:

Theorem 4. *$\mathcal{Q}$ is a weakly-modular orthocomplemented lattice.*

Axiom 9. *$\mathcal{Q}$ is separable; i.e., if $\{Q_\lambda\}$ is a family of mutually orthogonal nonzero questions, then $\{Q_\lambda\}$ is countable.*

Theorem 5. *$\mathcal{Q}$ is complete.*

Proof. This follows from Axiom 9 and the fact that every countable orthogonal family has an upper bound.

This is a good point at which to mention the question of superselection rules. A superselection rule is essentially a nontrivial observable which commutes with all observables. Thus we have superselection rules if and only if there exist questions different from zero and one which commute with all questions. An element which commutes with all elements of an orthocomplemented lattice L is said to be in the *center* of L.

Theorem 6. *Let L be a complete weakly-modular orthocomplemented lattice. Then the center of L is a sublattice of L and is a complete Boolean algebra.*

Proposition 10. Let E be an element in the center of an orthocomplemented lattice L. Assume that E is not equal to zero or one. Then L is the direct sum of [O,E] and [O,E'], where [O,E] is the lattice of questions Q such that $Q \leq E$.

An orthocomplemented lattice L is said to be *irreducible* if it cannot be written as the direct sum of two orthocomplemented lattices. From a mathematician's point of view, it is natural to ask if the study of arbitrary, complete, weakly-modular, orthocomplemented lattices can be reduced to the study of the irreducible ones. One might hope to do this by a procedure like von Neumann's direct integral decomposition of rings of operators. However, it is not known that such a procedure exists in the lattice setting. The only case that can be handled at present is a relatively trivial one. To discuss this case, we introduce some terminology.

For elements P and Q in a partially-ordered set, we say that P *covers* Q if $P > Q$ and there does not exist R with $P > R > Q$. P is an *atom* if P covers O. A lattice is *atomic* if every element is the join (perhaps infinite) of atoms.

Theorem 7. Let L be a complete weakly-modular, orthocomplemented lattice whose center is atomic. Then L is a direct sum $L = \Sigma \oplus L_i$ of irreducible orthocomplemented lattices.

Corollary. If L is a complete, atomic, weakly-modular, orthocomplemented lattice, then L is a direct sum of irreducible, atomic, orthocomplemented lattices.

V. A REPRESENTATION THEOREM

The property that $\mathcal{Q}$ is atomic may be the essential difference between $[\mathcal{O}(H), \mathcal{S}(H)]$ and more general systems which might be of physical interest. Naturally, it would be desirable to carry out more analysis without making the assumption of atomicity. Unfortunately, there seems, at the moment, to be no way to obtain a useful representation theorem for $\mathcal{Q}$, or $\mathcal{O}$, in the nonatomic case. One part of the difficulty is the lack of a theorem analogous to Theorem 7, representing arbitrary $\mathcal{Q}$ in terms of irreducible ones. However, even for irreducible $\mathcal{Q}$, there are great problems. To see this, one has only to consider the variety of weakly-closed Jordan algebras of self-adjoint operators in Hilbert space. The set of projections in any such algebra may be taken as a $\mathcal{Q}$ satisfying the axioms given so far. Obviously, such variety works against our finding a representation theorem. On the other hand, if the Jordan algebra is such that $\mathcal{Q}$ is atomic, then $\mathcal{Q}$ is essentially the lattice of all closed subspaces. For these reasons, we now introduce:

Axiom 10. $\mathcal{Q}$ is atomic.

In this form, the axiom appears nonphysical, but it does have a physical interpretation. We could replace Axiom 10 by the following (non-equivalent) axiom.

*Axiom 10'. a) The state ϕ in Axiom 6, such that $m_\phi(Q) = 1$, may be chosen as a pure state.**

b) If ϕ is a pure state, then there exists a question Q, such that $m_\psi(Q) = 1$, if and only if $\phi = \psi$.

The second half of this axiom is really an assertion that pure states may be realized in the laboratory.

Proposition 11. Axiom 10' implies Axiom 10.

Now, because of Theorem 7, we may focus our attention on the irreducible $\mathcal{Q}$. (It is easily verified that if $\mathcal{Q}$ is a direct sum $\mathcal{Q} = \Sigma \oplus \mathcal{Q}_j$, then each $\mathcal{Q}_j$ also satisfies our axioms.) Therefore we assume:

Axiom 11. $\mathcal{Q}$ is irreducible; i.e., there exist no nontrivial questions that commute with all other questions.

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*A pure state is one that cannot be written as a convex combination of other states.

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The property that Q is atomic may be the essential difference between $[Q(H), S(H)]$ and more general systems which might be of physical interest. Naturally, it would be desirable to carry out more analysis without making the assumption of atomicity. Unfortunately, there seems, at the moment, to be no way to obtain a useful representation theorem for Q, or $\mathcal{E}$, in the nonatomic case. One part of the difficulty is the lack of a theorem analogous to Theorem 7, representing arbitrary Q in terms of irreducible ones. However, even for irreducible Q, there are great problems. To see this, one has only to consider the variety of weakly-closed Jordan algebras of self-adjoint operators in Hilbert space. The set of projections in any such algebra may be taken as a Q satisfying the axioms given so far. Obviously, such variety works against our finding a representation theorem. On the other hand, if the Jordan algebra is such that Q is atomic, then Q is essentially the lattice of all closed subspaces. For these reasons, we now introduce:

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Axiom 11. Q is irreducible; i.e., there exist no nontrivial questions that commute with all other questions.

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*A pure state is one that cannot be written as a convex combination of other states.

The natural mathematical question to ask at this point is, "What can $\mathcal{Q}$ be?" It may be that the axioms given so far imply that $\mathcal{Q}$ is the lattice of closed subspaces of some Hilbert space, although over what number field is certainly left open. However, all we can do at the present time is indicate how further axioms can be introduced to assure the desired conclusion.

The next axiom is a regularity assumption. To a mathematician, the most regular lattices are those, such as Boolean algebras, in which the distributive laws $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$ and $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$ hold. A weaker regularity assumption is that the modular law holds; i.e., $X \leq Z$ implies $(X \cup Y) \cap Z = X \cup (Y \cap Z)$ for all Y. Birkhoff and von Neumann, in their original study of quantum logic,[1] assumed that the modular law holds. On the other hand, the modular law does not hold for $\mathcal{Q}(H)$. This was one of von Neumann's motivations for studying rings of operators and continuous geometries, for there one finds orthocomplemented lattices that are modular but not finite-dimensional in the ordinary sense. So far, these lattices do not seem to be important in quantum mechanics, but the question is very much an open one. Birkhoff[2] remarks that several concrete examples must be worked out before the question of modularity or nonmodularity can hope to be resolved. In particular, he mentions the sublattice of $\mathcal{Q}(H)$ generated by the characteristic functions of position and momentum observables. This by itself is a plausible model for many quantum mechanical problems; yet its properties are not known.

The failure of the modular law for $\mathcal{Q}(H)$ does not by any means signify that $\mathcal{Q}(H)$ is pathological. In fact, it is very close to being modular, and many of the manipulations characteristic of modular lattices may be applied when working with $\mathcal{Q}(H)$. The regularity law that does hold in $\mathcal{Q}(H)$ is that of semimodularity. This is defined by using the notion of a *modular pair*, which is a pair of elements Y,Z in a lattice such that, whenever $X \leq Z$, we have $(X \cup Y) \cap Z = X \cup (Y \cap Z)$. A lattice is *semimodular* if the relation of being a modular pair is symmetric. We can similarly define the terms *d-modular pair* (short for dual-modular pair) and *dual semimodular*. In an orthocomplemented lattice, which is, of course, self-dual, semimodularity and dual semimodularity are equivalent. The physical meaning of semimodularity, if it has one at all, is not clear. However, $\mathcal{Q}(H)$ is semimodular, and in our situation, semimodularity is equivalent to some simpler conditions.

Theorem 8. Let L be an atomic orthocomplemented lattice. Then the following statements about L are equivalent:

1) L is semimodular.

2) If P and Q cover $P \cap Q$, then $P \cup Q$ covers P and Q.

3) If P is an atom, and $P \cap Q = 0$, then $P \cup Q$ covers Q.

Further, if L is weakly-modular, condition 2) may be replaced by:

2') If P and Q are distinct atoms, then $P \cup Q$ covers P and Q.

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2') If P and Q are distinct atoms, then $P \cup Q$ covers P and Q.

Let us assume, with the rather feeble justification that this is a weak regularity hypothesis:

Axiom 12. The lattice $\mathcal{Q}$ is semimodular.

Now we can get a useful representation theorem. It involves the notion of what we call a semi-inner product space. Let R be a division ring, i.e., a not-necessarily commutative field, and let V be a left vector space over R; that is, in V we have addition and multiplication on the left by scalars from R. A semibilinear functional B on V is a map $(x,y) \to B(x,y)$ from V x V into R such that:

1) For all x_1, x_2, y_1, and y_2 in V and α in R,

   $B(\alpha x_1 + x_2, y_1 + y_2) = \alpha B(x_1,y_1) + \alpha B(x_1,y_2) + B(x_2,y_1) + B(x_2,y_2)$;

   and

2) There exists an antiautomorphism θ of R such that, for all x and y in V and α in R,

   $B(x,\alpha y) = B(x,y)\theta(\alpha)$.

It is important to note that θ is an antiautomorphism, i.e., $\theta(\alpha\beta) = \theta(\beta)\theta(\alpha)$, and that the multiplication of $B(x,y)$ by $\theta(\alpha)$ is on the right. Of course, this does not matter if the field is commutative.

We say that a semibilinear functional B is a *semi-inner product* if it satisfies the following conditions:

1) The antiautomorphism θ associated with B is involutory.

2) $B(x,y) = \theta[B(y,x)]$.

3) $B(x,x) = 0$ implies $x = 0$.

4) For some x, $B(x,x) = 1$.

We call a left vector space V over R, together with a semi-inner product, a *semi-inner product space*. If X is a subspace of such a space, we let $X^\perp$ denote the subspace of all y such that $B(x,y) = 0$ for all x in X. We say that X is *closed* if $X = X^{\perp\perp}$. Note that in an ordinary inner product space, this is not necessarily the same as topological closure.

Theorem 9. Let V be a semi-inner product space. Then the lattice L(V) of all closed subspaces of V is a complete, irreducible, atomic, semimodular, orthocomplemented lattice.

Theorem 10. Let L be a complete, atomic, irreducible, semimodular, orthocomplemented lattice of dimension at least four (i.e., there exist at least four mutually orthogonal atoms). Then there exists a semi-inner product space V such that L is orthoisomorphic to L(V).

This representation theorem (from MacLaren[24]) was obtained by extending a combination of several older results. In particular, the connection between a semibilinear form and orthocomplementation in finite-dimensional

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1) For all x_1, x_2, y_1, and y_2 in V and α in R,

$$B(\alpha x_1 + x_2, y_1 + y_2) = \alpha B(x_1,y_1) + \alpha B(x_1,y_2) + B(x_2,y_1) + B(x_2,y_2),$$

and

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<u>Theorem 9.</u> *Let V be a semi-inner product space. Then the lattice $L(V)$ of all closed subspaces of V is a complete, irreducible, atomic, semimodular, orthocomplemented lattice.*

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This representation theorem (from MacLaren[25]) was obtained by extending a combination of several older results. In particular, the connection between a semibilinear form and orthocomplementation in finite-dimensional vector spaces was established by Birkhoff and von Neumann.[1] Piron[12] has obtained a representation theorem in terms of subspaces of a projective geometry. Zierler[22] considered (see below) the special case where the lattice under a finite-dimensional element was the lattice of subspaces of a real or complex vector space, and then showed that the whole of $\mathcal{Q}$ was a lattice of subspaces of an inner product space.

The representation theorem certainly brings us close to the conclusion that $\mathcal{Q}$ is $\mathcal{Q}(H)$ for either a real, complex, or quaternionic Hilbert space H; and this is one of the things that make the quantum logic approach attractive. However, certain highly nontrivial problems remain to be solved. For one thing, we must show that V, assuming it is actually an inner product space, is complete in the usual norm topology. It may well be that this can be handled by a further innocuous axiom, or, even better, completeness may follow from the axioms we already have. In particular, the following conjecture may be true.

Conjecture. *Let* V *be an inner product space, and let* $L(V)$ *be the lattice of all subspaces* X *such that* $X = X^{\perp\perp}$. *Then, if* $L(V)$ *is weakly-modular,* V *is complete in the usual norm topology.*

A second, perhaps more difficult, problem is to prove that the division ring R is actually the real, complex, or quaternionic numbers.* The problem appears to be more serious. There are many possibilities besides the real, complex, or quaternionic numbers which yield lattices satisfying most of our axioms. It is the author's opinion, however, that the axioms given above imply that R is the field of real, complex, or quaternionic numbers. Axioms 6 and 7 (existence of sufficiently many states and of sums of bounded observables) seem important here.

VI. THE FINAL AXIOMS

To complete the set of axioms, such as it is, let us follow Zierler and introduce axioms that will characterize the division ring appearing in the representation theorem.

Axiom 13.** *Let* E *be a nonzero element of finite dimension in* $\mathcal{Q}$. *Then the set of all atoms* P *such that* $P \leq E$ *is compact in the norm topology.*

Axiom 14. *For some finite* E *in* $\mathcal{Q}$ *and real interval* I, *there exists a continuous nonconstant function* $t \to Q_t$ *from* I *to the lattice* $[0,E]$.

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*Moreover, one must then show that in the case of the complex numbers, the automorphism θ associated with the semi-inner product may be taken as the usual conjugation automorphism. Fortunately, Zierler has shown how this can be done. In his original paper,[22] there was some ambiguity on this point; but he clears this up in Ref. 23. It is worth noting that there exists an orthocomplementation of the lattice of subspaces of a finite-dimensional complex vector space which is not equivalent to the normal one (see MacLaren[24]). The two orthocomplemented lattices are not ortho-isomorphic, even though as lattices they are identical.

**Zierler (Ref. 22, p. 1162) assumes that, for each $n \leq \dim [E]$, the set of questions in $[0,E]$ of dimension n is compact. However, that follows easily from the axiom given here.

vector spaces was established by Birkhoff and von Neumann.[1] Piron [illegible] has obtained a representation theorem in terms of subspaces of a projective geometry. Zierler [illegible] considered (see [illegible]) the special case where the lattice under a finite-dimensional [illegible] ... the lattice of subspaces of a real or complex vector space, and then showed that the [illegible] was a lattice of subspaces of an inner product space.

The representation theorem [illegible] close to the conclusion that Q is L(H) for either a real, complex, or quaternionic Hilbert space H and this is one of the things that make the [illegible] attractive. However, certain [illegible] still to be solved. For one thing, we must show that V, assumed to be [illegible] inner product space, is complete in the usual norm [illegible] can be handled by a further [illegible] completeness may follow from the axioms we already have. In particular, the following conjecture may be true.

Conjecture. Let V be an inner product [illegible] and let L(V) be the lattice of all subspaces X such that $X = X^{\perp\perp}$. Then if L(V) is [illegible], V is complete in the usual norm [illegible].

A second, perhaps more difficult, problem is to prove that the division ring K is actually the real, complex, or quaternionic numbers.[*] The problem appears to be more serious. There are many [illegible] besides the real, complex, or quaternionic numbers which [illegible] of our axioms. It is the author's opinion, however, that the axioms given above imply that K is the field of real, complex, or quaternionic numbers. Axioms 6 and 7 (existence of sufficiently many states and [illegible] of bounded observables) seem important here.

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Axiom 11. For some [illegible] K is Q and [illegible]

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Now Zierler's results can be combined with Theorem 10 to conclude:

Theorem 11. If $\mathcal{Q}$ contains at least four orthogonal atoms, then $\mathcal{Q}$ is isomorphic to the lattice of all closed (in the sense that $X = X^{\perp\perp}$) subspaces of an inner product space H over the real, complex, or quaternionic numbers.

We leave it to the reader to decide whether an additional axiom is needed to insure that H is complete, and also what axioms should be introduced to guarantee that the division ring is the complex numbers, rather than the reals or quaternions.

All the above discussion has been about the problem of proving that $\mathcal{O}$ is $\mathcal{O}(H)$, or equivalently that $\mathcal{Q}$ is the lattice $\mathcal{Q}(H)$ of all closed subspaces of H; little has been said about S. However, once it is known that $\mathcal{Q}$ is $\mathcal{Q}(H)$, the exact nature of S is easily deduced. Gleason has shown that every measure on $\mathcal{Q}(H)$ is in $S(H)$. [$S(H)$ was defined in the Introduction as the measures coming from trace operators.] Thus, every measure m on $\mathcal{Q}(H)$ may be written as a convex combination, $m = \Sigma\alpha_i m_i$, where each m_i is a measure of the form $P \to (Px,x)$, x a unit vector defining m_i. Now it follows immediately from Axiom 6 that every measure m of the form $m(P) = (Px,x)$ is in S^*. Thus making the normal assumption that S^*, or equivalently S, is closed under countable convex combinations, we conclude that $S = S(H)$. For completeness, let us include that last axiom:

Axiom 15. The set of states S is closed under the taking of countable convex combinations.

This concludes our development of the axioms, but it may be worth making a few remarks about Axioms 13 and 14. Axiom 14 is relatively innocuous. It is almost physical; and, moreover, it can probably be dropped entirely with only moderate effort. It is only used to prove that the coordinatizing division ring R is not totally disconnected in the topology that it inherits from $\mathcal{Q}$.

Axiom 13, which is used to show that the coordinatizing division ring is locally compact, is more interesting. Although there is no apparent physical meaning to this axiom, there is a connection between it and the Jordan algebra approach. Consider the set $A(E)$ of observables with spectral decompositions $\Sigma\lambda_i F_i$, $F_i \leq E$ all i. If the whole algebra A of bounded observables is a Jordan algebra,† then $A(E)$ will also be a Jordan algebra. Suppose that $A(E)$ has a finite basis as a vector space. Then it is easy to verify that Axiom 13 holds. On the other hand, one can prove from Axiom 13 that $A(E)$ is a locally compact normed vector space, and hence that $A(E)$ has a finite basis. Thus Axiom 13 corresponds roughly to the finite basis condition for the Jordan algebra $A(E)$. Of course, given that $A(E)$ is a Jordan algebra, the nature of the division ring R could be deduced from the structure theorems on Jordan algebras in Ref. 8. Moreover, in this situation, it seems clear that Axiom 14 can be dropped.

†This means assuming the distributive law for the Jordan product in addition to our other axioms.

Now Piron's results can be combined with Theorem 14 to conclude:

Theorem 15. If Q contains at least four orthogonal atoms, then Q is isomorphic to the lattice Q_c of closed (in the sense that $X = X''$) subspaces of a Hilbert space H over the reals, complex, or quaternionic numbers.

We leave it to the reader to decide whether an additional axiom is needed to insure that H is complete, and also what axioms should be introduced to guarantee that the division ring is the complex numbers, rather than the reals or quaternions.

All the above discussion has been about the problem of proving that Q is $Q(H)$, or equivalently that Q is the lattice of all closed subspaces of H; little has been said about S. However, once it is known that Q is $Q(H)$, the exact nature of S is easily deduced. Gleason has shown that every measure on $Q(H)$ is in $S(H)$. [$S(H)$ was defined to be the intersection of the measures coming from trace operators.] Thus, every measure m on $Q(H)$ may be written as a convex combination, $m = \Sigma a_i m_i$, where each m_i is a measure of the form $P \to (Px_i,x_i)$, x_i a unit vector defining m_i. Now it follows immediately from Axiom 6 that every measure m of the form $m(P) = (Px,x)$ is in S^*. Thus making the natural assumption that S^*, or equivalently S, is closed under countable convex combinations, we conclude that $S^* = S(H)$. For completeness, let us include that last axiom:

Axiom 15. The set of states S is closed under the formation of countable convex combinations.

This concludes our development of the axioms, but it may be worth making a few remarks about Axioms 13 and 14. Axiom 14 is relatively innocuous. It is almost physical; and, moreover, it can probably be dropped entirely with only moderate effort. It is only used to prove that the coordinatizing division ring R is not totally disconnected in the topology that it inherits from Q.

Axiom 13, which is used to show that the coordinatizing division ring is locally compact, is more interesting. Although there is no apparent physical meaning to this axiom, there is a connection between it and the Jordan algebra approach. Consider the set $A(E)$ of observables with spectral decompositions $\{P_i\}$, $P_i \leq E$ all i. If the whole algebra A of bounded observables is a Jordan algebra,* then $A(E)$ will also be a Jordan algebra. Suppose that $A(E)$ has a finite basis as a vector space. Then it is easy to verify that Axiom 13 holds. On the other hand, one can prove from Axiom 13 that $A(E)$ is a locally compact normed vector space, and hence that $A(E)$ has a finite basis. Thus Axiom 13 corresponds exactly to the finite basis condition for the Jordan algebra $A(E)$. Of course, since all $A(E)$ is a Jordan algebra, the nature of the division ring R could be deduced from the structure theorems on Jordan algebras in Ref. 8. Moreover, in this formulation, it seems clear that Axiom 14 can be omitted.

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*This means assuming the distributive law for the Jordan product in addition to our other axioms.

SOME REFERENCES ON THE FOUNDATIONS OF QUANTUM MECHANICS

1. G. Birkhoff and J. von Neumann, *The Logic of Quantum Mechanics*, Ann. Math. *37* (1936), p. 823.
2. G. Birkhoff, *Lattices in Applied Mathematics*, Proceedings of a Symposium in Pure Mathematics *2*, Lattice Theory, Am. Math. Soc. (1961).
3. G. Emch, *Mechanique Quantique Quaternionienne et Relativite Restreinte*, thesis, University of Geneva.
4. G. Emch and C. Piron, *Symmetry in Quantum Theory*, J. Math. Phys. *4* (1963), p. 469.
5. D. Finkelstein, J. M. Jauch, S. Schiminorich, and D. Speiser, *Foundations of Quaternion Quantum Mechanics*, J. Math. Phys. *3* (1962), p. 207.
6. A. M. Gleason, *Measures on the Closed Subspaces of a Hilbert Space*, M. Rat. Mech. and Analysis *6* (1957), p. 885.
7. S. Gudder, *A Generalized Probability Model for Quantum Mechanics*, thesis, University of Illinois (1964).
8. P. Jordan, J. von Neumann, and E. Wigner, *On an Algebraic Generalization of the Quantum Mechanical Formalism*, Ann. Math. *35* (1934), p. 29.
9. D. B. Lowdenslager, *On Postulates for General Quantum Mechanics*, Proc. Am. Math. Soc. *8* (1957), p. 88.
10. G. W. Mackey, *Quantum Mechanics and Hilbert Space*, Amer. Math. Monthly *64* (1957), p. 45.
11. G. W. Mackey, *Mathematical Foundations of Quantum Mechanics*, W. A. Benjamin, New York (1963).
12. C. Piron, *Axiomatique Quantique*, thesis, Lausanne University (1963).
13. J. C. T. Pool, *Simultaneous Observability and the Logic of Quantum Mechanics*, thesis, State University of Iowa (1963).
14. I. E. Segal, *Mathematical Problems of Relativistic Physics*, Am. Math. Soc., Providence (1963).
15. I. E. Segal, *Postulates for General Quantum Mechanics*, Ann. Math. *48* (1947), p. 930.
16. S. Sherman, *On Segal's Postulates for General Quantum Mechanics*, Ann. Math. *64* (1956), p. 593.
17. S. Sherman, *Non-negative Observables Are Squares*, Proc. Am. Math. Soc. *2* (1951), p. 31.
18. D. M. Topping, *Jordan Algebras of Self-adjoint Operators*, Memoirs Am. Math. Soc. *53* (1965).
19. V. S. Varadarajan, *Probability in Physics and a Theorem on Simultaneous Observability*, Comm. Pure and App. Math. *15* (1962), pp. 189-217.
20. J. von Neumann, *The Mathematical Foundations of Quantum Mechanics*.
21. J. von Neumann, *On an Algebraic Generalization of the Quantum Mechanical Formalism* (Part 1), Mat. Sb. III (1936), p. 415; also in collected works, Vol. 3.

SOME REFERENCES ON THE FOUNDATIONS OF QUANTUM MECHANICS

1. G. Birkhoff and J. von Neumann, *The Logic of Quantum Mechanics*, Ann. Math. *37* (1936), p. 823.

2. G. Birkhoff, *Lattices in Applied Mathematics*, Proceedings of a Symposium in Pure Mathematics 2, Lattice Theory, Am. Math. Soc. (1961).

3. G. Emch, *Mécanique Quantique Quaternionienne et Relativité Restreinte*, thesis, University of Geneva.

4. G. Emch and C. Piron, *Symmetry in Quantum Theory*, J. Math. Phys. *4* (1963), p. 469.

5. D. Finkelstein, J. M. Jauch, S. Schiminovich, and D. Speiser, *Foundations of Quaternion Quantum Mechanics*, J. Math. Phys. *3* (1962), p. 207.

6. A. M. Gleason, *Measures on the Closed Subspaces of a Hilbert Space*, J. Rat. Mech. and Analysis *6* (1957), p. 885.

7. S. Gudder, *A Generalized Probability Model for Quantum Mechanics*, thesis, University of Illinois (1964).

8. P. Jordan, J. von Neumann, and E. Wigner, *On an Algebraic Generalization of the Quantum Mechanical Formalism*, Ann. Math. *35* (1934), p. 29.

9. D. B. Lowdenslager, *On Postulates for General Quantum Mechanics*, Proc. Am. Math. Soc. *8* (1957), p. 88.

10. G. W. Mackey, *Quantum Mechanics and Hilbert Space*, Amer. Math. Monthly *64* (1957), p. 45.

11. G. W. Mackey, *Mathematical Foundations of Quantum Mechanics*, W. A. Benjamin, New York (1963).

12. C. Piron, *Axiomatique Quantique*, thesis, Lausanne University (1963).

13. J. C. T. Pool, *Simultaneous Observability and the Logic of Quantum Mechanics*, thesis, State University of Iowa (1963).

14. I. E. Segal, *Mathematical Problems of Relativistic Physics*, Am. Math. Soc., Providence (1963).

15. I. E. Segal, *Postulates for General Quantum Mechanics*, Ann. Math. *48* (1947), p. 930.

16. S. Sherman, *On Segal's Postulates for General Quantum Mechanics*, Ann. Math. *64* (1956), p. 593.

17. S. Sherman, *Non-negative Observables Are Squares*, Proc. Am. Math. Soc. *2* (1951), p. 31.

18. D. M. Topping, *Jordan Algebras of Self-adjoint Operators*, Memoirs Am. Math. Soc. *53* (1965).

19. V. S. Varadarajan, *Probability in Physics and a Theorem on Simultaneous Observability*, Comm. Pure and App. Math. *15* (1962), pp. 189-217.

20. J. von Neumann, *The Mathematical Foundations of Quantum Mechanics*.

21. J. von Neumann, *On an Algebraic Generalization of the Quantum Mechanical Formalism (Part I)*, Mat. Sb. III (1936), p. 415; also in collected works, Vol. 3.

22. N. Zierler, *Axioms for Non-relativistic Quantum Mechanics*, Pacific J. Math. *II* (1961), p. 1151.

23. N. Zierler, *On the Lattice of Closed Subspaces of Hilbert Space*, Technical Memorandum, TM-04172/0000/00/0/00, Mitre Corporation, Bedford, Mass. (1965).

24. M. D. MacLaren, *Atomic Orthocomplemented Lattices*, Pacific J. Math. *14* (1964), pp. 597-612.

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